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On two-dimensional laminar wakes and jets

By KARL K. CHEN
Department of the Aerospace and Mechanical Engineering Sciences,
University of California, San Diego†

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The flow fields in two-dimensional, isoenergetic, viscous free mixing with constant β and with initial velocity profiles deviating slightly from those given by wake-like solutions of the Falkner–Skan equation for that β are considered. The similar solutions of the Falkner–Skan equation are investigated in more detail than in the past, e.g. we show that as $\beta \to -1$ the flows approach the pure jet with the surrounding fluid at rest, and that there are new branch solutions for $\beta < -1$. We have investigated the spatial stability of these flows; it is found that for $\beta > -0{\cdot}5$ the only spatially stable solutions are the trivial ones $f'(\eta) \equiv 1$, but for $-1 < \beta < -0{\cdot}5$ there are non-trivial, jet-like solutions which are spatially stable. As to the new branch solutions for $\beta < -1$, all are spatially unstable.

1. Introduction

Wake-like solutions of the Falkner–Skan equation $f''' + ff'' + \beta(1 - f'^2) = 0$ satisfying the boundary conditions $f(0) = f''(0) = 0$, $f'(\infty) = 1$ in the range $-0{\cdot}5 < \beta < 0$ were first investigated by Stewartson (1954) in connexion with his suggestion that the fluid downstream of a point of separation might be considered to be bounded by a free stream. Lees & Reeves (1964) have suggested use of this family of solutions to study the base flow of a blunt body at supersonic speeds by 'unhooking' the profiles from the Falkner–Skan parameter β and selecting $f'(0) = u(0)/u_e$ as an independent parameter. Applying these wake-like solutions to study the flow immediately downstream of a flat-based body of finite dimension normal to the stream, Kennedy (1964) has shown that the solution of this family for $\beta \to 0$ approaches Chapmen's solution (with a shift of origin) for the constant pressure flow past an infinite step. Steiger & Chen (1965) have calculated jet-like solutions for the range $-1 < \beta < -0{\cdot}5$ and some wake-like solutions in the range $\beta > 1$.

Our purpose here is twofold. First, we provide some additional information concerning the similarity solutions. In particular, we give new solutions for $\beta < -1$. In addition, we show that the solution branch investigated by Stewartson approaches asymptotically the case of a 'pure' jet with the surrounding fluid at rest as $\beta \to -1^+$.

Our second purpose relates to flows which represent small departures from strictly similar flows of the wake and jet type. This follows the work of Libby & Fox (1963) and of Chen & Libby (1968), wherein linearized deviations from similar

† Present address: Avco Systems Division, Wilmington, Mass.

solutions are shown to lead to interesting and useful results for boundary layers. As one application of this extension, we cite the question of the spatial stability of these flows. Chen & Libby (1968) have discussed the spatial stability of the Falkner–Skan equation subject to the boundary-layer conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$, and have encountered an eigenvalue problem which leads to infinite sets of eigenvalues. Similarly, we consider here an initial value problem, i.e. we choose flows which are bounded by a free stream with a constant streamwise pressure gradient parameter β. At some streamwise station an initial velocity profile deviating slightly from that given by a similar solution is assumed to be specified. We investigate the downstream behaviour of such a wake or jet. If the similarity profile for that given β is approached as the streamwise distance from the initial station increases indefinitely, we consider the flow to be spatially stable and expect that such a flow could be observed in an appropriate experiment. We have investigated both the spatial stability of the similar solutions calculated by the previous authors and that of the trivial solution, i.e. $f'(\eta) \equiv 1$.

2. Analysis for similar flows

The two-dimensional isoenergetic, symmetric viscous free mixing with streamwise pressure gradient under conditions of similarity and of certain simplified thermodynamic and transport properties (cf. Steiger & Chen 1965) can be described by

$$f''' + ff'' + \beta(1-f'^2) = 0. \tag{1}$$

This, subject to the boundary conditions:

$$f(0) = f''(0) = 0, \quad f'(\infty) = 1 \quad \text{exponentially}, \tag{2}$$

where

$$\left.\begin{aligned} \eta = u_e/(2s)^{\frac{1}{2}}\int_0^y \rho\, dy, \quad s = \int_0^x \rho_e \mu_e u_e\, dx, \quad u = u_e(s) f'(\eta), \\ \beta = (2s/u_e)\,(du_e/ds)\,(H_e/h_e) = \text{constant}. \end{aligned}\right\} \tag{3}$$

Standard notation is used throughout.

The numerical solutions of (1) subject to the boundary conditions (2) were first investigated by Stewartson (1954), in the range $-0{\cdot}5 \leqslant \beta \leqslant 0$. The solutions in the same range were recalculated by Kennedy (1964) in his study of laminar flow behind flat-based bodies. Stewartson (1964) suggested an analytic relation which involves β and $f'(0)$ in the region $\beta \approx 0$, and which agrees reasonably well with the numerical results of Kennedy. Steiger & Chen (1965) presented the solutions of the same problem in the range $-1{\cdot}0 < \beta \leqslant -0{\cdot}5$ and $\beta > 1{\cdot}0$. The numerical results by previous authors are shown in curves B and C of figure 1 as $f'(0)$ as a function of β and are listed in table 1.

We now indicate that as $\beta \to -1^+, f'(0) \to \infty$ and that the limiting flow corresponding to $\beta = -1, f'(0) \to \infty$ is the two-dimensional jet in a surrounding fluid at rest. We follow the idea of Steinheuer (1968), who demonstrated that one solution branch of the Falkner–Skan equation for boundary layers approaches the solution for the wall jet.

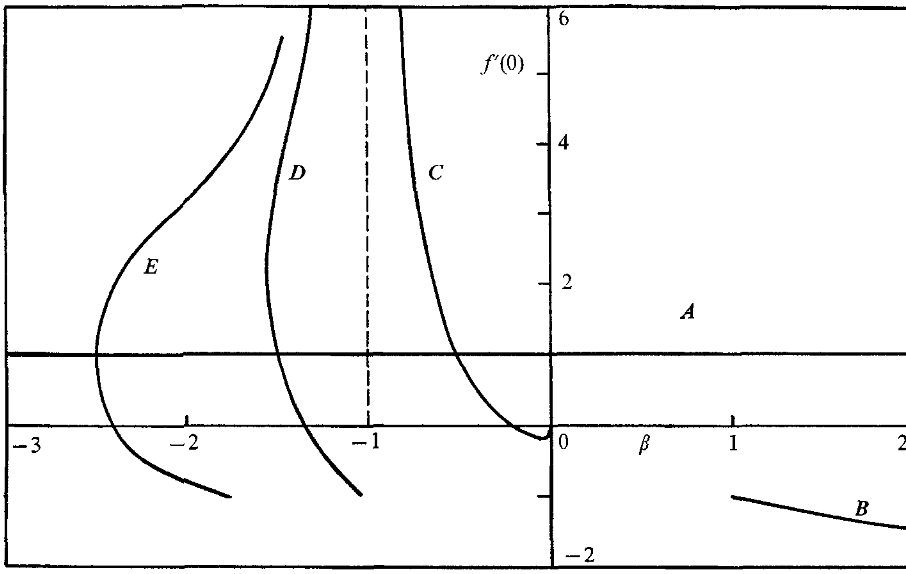

FIGURE 1. The variation of free streamline velocity with β.

$f'_{\infty}(0)$	β	λ_1
∞	$-1{\cdot}0$	0
100·0	−0·9853	0·0097
50·0	−0·9712	0·0188
20·0	−0·9321	0·0431
15·0	−0·9123	0·0569
10·0	−0·8760	0·0702
5·0	−0·7881	0·1110
2·5	−0·6705	0·1333
1·718	−0·6	0·1135
1·1	−0·5170	0·0308
1·0	−0·5	0·0
0·9	−0·4816	−0·0298
0·5370	−0·4	−0·4078
0·3644	−0·35	−1·0076
0·2210	−0·30	−2·7245
0·1015	−0·25	−11·79

TABLE 1. The free streamline velocity and the lowest eigenvalues with β for the usual solution branch, i.e. branch C.

The demonstration is straightforward. Substitution of the transformations

$$F(\xi) = Af(\eta), \quad \eta = A\xi \tag{4}$$

into (1) yields
$$F''' + FF'' + \beta(A^4 - F'^2) = 0, \tag{5}$$

subject to the boundary conditions

$$F(0) = F''(0) = 0, \quad F'(\infty) = A^2,$$

where the primes now denote differentiation with respect to ξ.

The previous function $f(\eta)$ and its derivatives are given in terms of the new function $F(\xi)$ as

$$f = (1/A)F, \quad f' = (1/A^2)F', \quad f'' = (1/A^3)F'', \quad f''' = (1/A^4)F'''.$$

With no loss of generality we may let $F''(0) = 1$ and determine numerically the relation between A and β from (5). We find numerically that as $\beta \to -1^+$, $A \to 0$. If we accept this behaviour as plausible, the limiting case $\beta = -1$, $f'(0) \to \infty$ for (1) is equivalent to that of $\beta = -1$ as $A \to 0$ for (5), which then becomes identical to the form for a two-dimensional jet with surrounding fluid at rest (cf. Schlichting 1960, pp. 164–168).

There are trivial solutions for (1) subject to the boundary condition (2), i.e. $f' \equiv 1$ for all β. Such solutions correspond to two contiguous, uniform streams subject to a streamwise pressure gradient. We mention them here because we use them as basic solutions in the later discussion.

Along the lines of Libby & Liu (1967), we can find new branch solutions which intersect once, or more than once, with the line $f' = 1$. The physical significance of these new branch solutions depends on a suitable initial profile, i.e. a profile with velocity overshoot or undershoot near the outer edge of a boundary layer, and a suitable external stream corresponding to a constant value of β. The numerical analysis used here is essentially the same as that described by Libby & Chen (1966), an application to the boundary-layer problem of the quasi-linearization technique of Bellman and Kabala, including treatment of the 'infinity' condition such that exponential decay is assured. Because of the great sensitivity of $f'(0)$ to β for the solution branches for $\beta < -1$, it has been found more convenient to specify a value for $f'(0)$ and to consider β as a parameter to be determined in each iteration cycle of the quasi-linear scheme. When the ratio of the difference of two successive values of β to β itself is within a specified tolerance, say 10^{-4}, we consider convergence to have been achieved. We have found the new solution branches denoted by D and E in figure 1 and have listed some typical prime $f'(0)$, β in table 2.

Before discussing the numerical results, we note that the values of β, yielding $f'(0) = 1$ on curves C, D and E, may be determined analytically as follows. Substitution of $f' = 1 + \tilde{f}_1'$ with $|\tilde{f}_1'| \ll 1$ into (1) and proper ordering of the results leads to

$$\tilde{f}_1''' + \eta \tilde{f}_1'' - 2\beta \tilde{f}_1' = 0, \tag{6}$$

subject to the boundary conditions

$$\tilde{f}_1(0) = \tilde{f}_1''(0) = 0, \quad \tilde{f}_1'(\infty) = 0.$$

It is advantageous to make the transformation

$$H = \tilde{f}_1' \exp(\eta^2/2).$$

The new dependent variable is found to satisfy the Hermite equation

$$H'' - \eta H' - (2\beta + 1)H = 0, \tag{7}$$

subject to the boundary conditions

$$H'(0) = 0, \quad \lim_{\eta\to\infty} H \exp(-\eta^2/2) = 0.$$

The eigenvalues, which have been discussed in detail by Pauling & Wilson (1935, p. 71), are

$$\beta = -0{\cdot}5, \ -1{\cdot}5, \ -2{\cdot}5, \ -3{\cdot}5, \ \ldots. \tag{8}$$

These values represent the intersection of solution branches with the trivial one $f'(0) \equiv 1$.†

$f'_\infty(0)$	β (branch D)	β (branch E)
−1·0	−1·0	−1·75
−0·9	−1·074	−1·925
−0·7	−1·151	−2·151
−0·5	−1·217	−2·264
0	−1·347	−2·415
0·5	−1·438	−2·482
1·0	−1·5	−2·500
2·0	−1·555	−2·400
2·5	—	−2·269
3·0	−1·531	−2·083
3·5	—	−1·882
4·0	−1·454	−1·714
4·5	—	−1·589
5·0	—	−1·47
6·0	−1·306	—
8·0	−1·220	—
10·0	−1·171	—
12·0	−1·139	—
19·0	−1·135	—

TABLE 2. The free streamline velocity with β for the first new solution branch, (branch D) and for the second new solution branch (branch E).

We now return to our numerical results and the new solution branches. First we comment on agreement with other results. The values of β yielding $f'(0) = 0$ in figure 1 are identical to those yielding $f''_w = 0$ in the analysis of Libby & Liu (1967), since they satisfy the same equation and the same boundary conditions in this special case. In addition, the numerical calculations for $f'(0) \gtrsim 1$ check the analytic prediction of β given by (18).

The new branches are of limited extent. For example, on branch D as $f'(0)$ increases beyond $f'(0) = 3{\cdot}5$, where $\beta = -1{\cdot}49$, $f' < 0$ as shown in figure 2. The extent of the range of η for which $f'(\eta) < 0$ increases as $f'(0)$ increases along this branch until it reaches the point $f'(0) = 19$, $\beta = -1{\cdot}135$, beyond which $f'(\eta) < -1$ would occur at some η, say η^*. But Stewartson (1953) pointed out that it is impossible to have $f'^2(\eta) > 1$, $f''(\eta) = 0$, and $f'''(\eta) > 0$ for a negative β, so that f'

† A referee has called our attention to the fact that if asymmetric flows close to uniformity are admitted, then negative integer values of β in the sequence (8) are possible. Throughout the present work, only symmetric flows are considered.

cannot have a minimum when regarded as a function of η in a range yielding $f'^2 > 1$. Once $f' < -1$ occurs at some η, f' can never turn back to satisfy the outer boundary condition, i.e. $f'(\infty) = 1$. Thus, we suppose that there exists no solution on this branch for $f'(0) \gtrsim 19$.

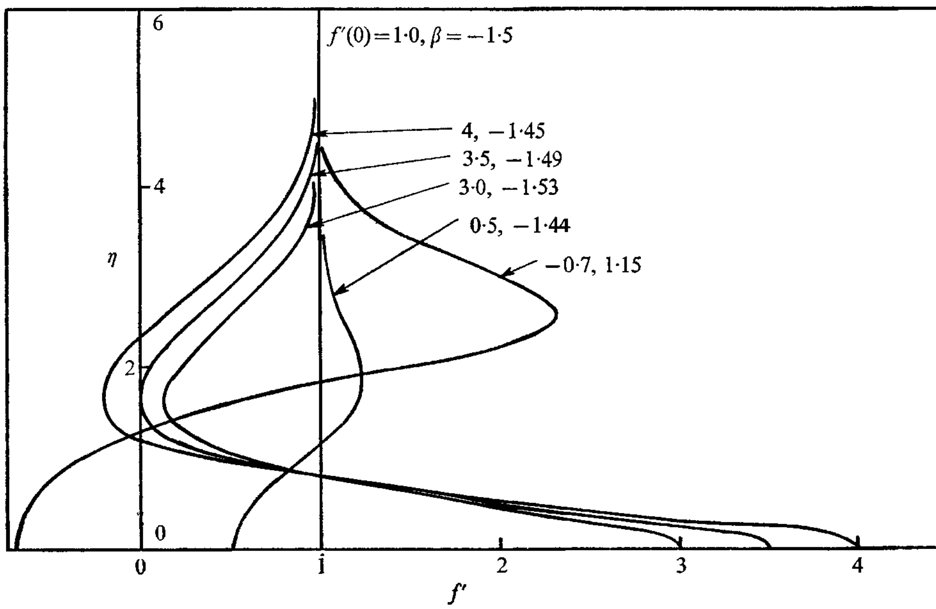

FIGURE 2. Typical velocity profiles for the first new solution branch, i.e. branch D.

The same argument applies to the other end of this branch, i.e. below $f'(0) = -1$. Numerical calculation does not permit the exact value of β, corresponding to $f'(0) = -1$, but extrapolation indicates that the terminal point is probably $\beta = -1$.

The second new solution branch, i.e. of curve E, appears to be similar to the first one, i.e. $f' < 0$ occurs at some η as $f'(0)$ increases beyond the point ($f'(0) = 2\cdot4, \beta = -2\cdot30$), while the branch will terminate at the points ($f'(0) = 5\cdot0$, $\beta = -1\cdot47$) and ($f'(0) = -1$, $\beta = -1\cdot75$).

Before leaving these similarity solutions, we remark on the solutions for $0 < \beta < 1$. It is interesting that, in this range, only the case $f'(\eta) \equiv 1$ appears to exist. There seems to be no theoretical reason why this is so for general β. However, for the special case of $\beta = \frac{1}{2}$, we can prove that for $f'(0) \geqslant -1$ no solution can exist. Moreover, although we have tried by various techniques, we have been unable to find numerically any solutions in this range of β.

Presumably, there are further new solution branches to the left of curve E, but we do not investigate them here. One of the questions relevant to new solution branches such as those found here concerns their physical significance. In studying this question, we may follow the spatial stability argument of Chen & Libby (1968). Accordingly, consider a particular similar flow for $\beta < 0$. Since from (3) we know that strict similarity cannot apply for all s if $\beta < 0$, we imagine that similarity can apply for $s > s_0$, i.e. downstream of some initial length in which the similarity profile corresponding to the specified β is established. Now suppose there are in this initial profile small departures from strict similarity; the question to be answered is whether these deviations grow with increasing s or decay so that, as $s \to \infty$, the similarity solution is achieved. If the former situation prevails, the flow is said to be spatially unstable and is considered to be physically unrealizable, except possibly as a local state in a completely non-similar flow. On the contrary, if the flow is spatially stable we would expect it could be observed in a suitable experiment.

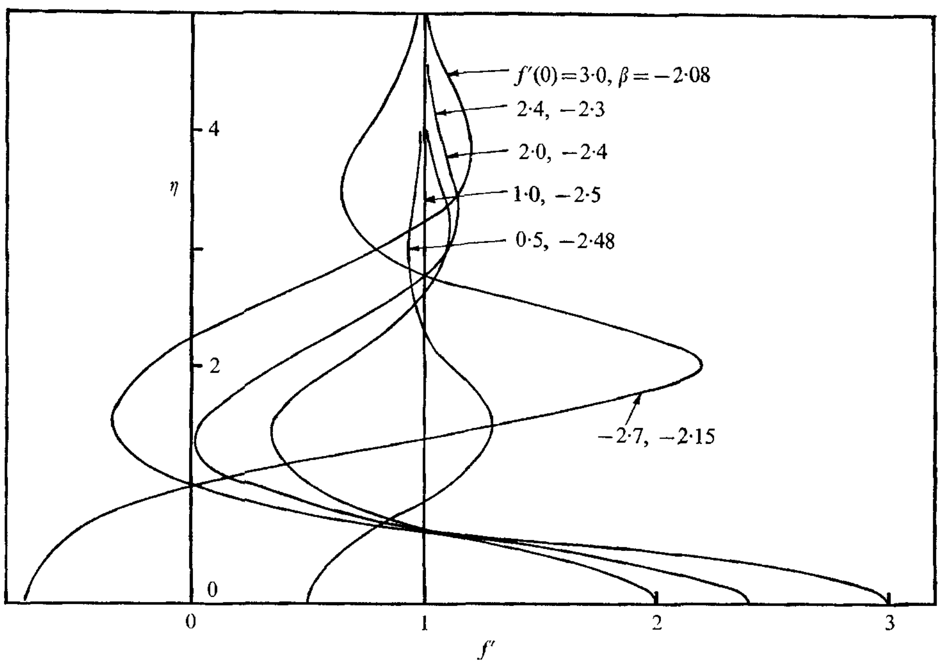

FIGURE 3. Typical velocity profiles for the second new solution branch, i.e. branch *E*.

The above considerations are in one sense restricted versions of those made for boundary layers with $\beta \geqslant 0$ by Serrin (1967), who showed that a boundary layer with any arbitrary initial profile approaches the relevant similar flow as $s \to \infty$; in another sense they are more general versions in that any β pertains.

We now proceed to examine the spatial stability of the previously obtained and new solutions to the Falkner–Skan equation for wake-like and jet-like flow. We remark that the eigenfunctions and eigenvalues arising in the study of spatial stability have applicability to other analysis.

3. Analysis of nearly similar flows

If we follow the analysis of Chen & Libby (1968), it is readily found that the perturbations about a similar solution $f(\eta)$ corresponding to a particular β may be described by the eigenfunction $N_n(\eta)$ and related eigenvalues λ_n, defined by

$$N_n''' + fN_n'' + (\lambda_n - 2\beta) f' N_n' + (1-\lambda_n) f'' N_n = 0, \tag{9}$$

$$N_n(0) = N_n''(0) = 0 \quad \text{and} \quad N_n'(\infty) = 0, \quad \text{exponentially.}$$

Moreover, following their work, we can show that the eigenvalues are real, and the eigenfunctions orthogonal in the sense

$$\int_0^\infty f'^4 \left(\exp \int_0^\eta f\,d\eta\right) (N_n/f')' \,(N_m/f')'\,d\eta = \delta_{nm} C_n. \tag{10}$$

Finally, we can examine whether the eigenfunctions are positive. Again following Chen & Libby (1968), we must examine the cases $\beta \geqslant 0$ and $\beta < 0$, and the cases wherein $f'(\eta)$ is either positive definitely or negative in some range of η must be treated separately. For any β if $f'(\eta)$ is piecewise negative, then infinite sequences of positive and negative λ's exist. For $\beta > 0, f(\eta) > 0$, the λ's are all positive; for $\beta < 0$, $f(\eta) > 0$, there exists a sequence of increasing λ's, but the lowest one need not be positive.

With these facts established, it is possible to employ numerical analysis to determine for a given $f(\eta)$ and corresponding β the eigenfunctions and eigenvalues. For example, the work of Libby & Chen (1968), which we have applied to the present flows, shows how quasi-linearization extended to eigenvalue problems may be so used.

The eigenvalue problem for the special case of the flows $f' \equiv 1$ can be studied analytically. On physical grounds, this case corresponds to the examination of the downstream effect of a small, symmetric departure from a uniform stream subject to a streamwise pressure gradient corresponding to similarity. Substitution of $f = \eta$, $f' \equiv 1$ and $f'' \equiv 0$ into (9) yields

$$N''' + \eta N'' + (\lambda - 2\beta) N' = 0, \tag{11}$$

subject to the boundary conditions (9). Equation (11) is of the same form as (7) and yields the eigenvalues

$$\lambda_n = 2\beta + 1,\ 2\beta + 3,\ 2\beta + 5,\ \ldots,\ 2\beta + 2n - 1,\ \ldots. \tag{12}$$

Pauling & Wilson (1935) list the corresponding eigenfunctions, which are orthogonal in a sense consistent with (10), provided $f' \equiv 1$ and $f = \eta$.

The lowest eigenvalues, $\lambda_1 = 2\beta + 1$, which we show in figure 4, are positive for $\beta > -0{\cdot}5$. This result implies within this linear theory that a boundary layer with an initial profile close to the trivial solution, $f' \equiv 1$, for the pressure gradient parameter β greater than 0·5 will approach uniformity with increasing downstream distance. For the range $\beta < -0{\cdot}5$ the trivial solutions of (1) are unstable to arbitrary perturbations in initial data.

However, it is clear from the sequence in (12) that, for $-1{\cdot}5 < \beta \leqslant -0{\cdot}5$, the uniform flow is stable to all perturbations except those related to the first eigenfunction, $N_1(\eta)$, and that similar considerations apply to successive ranges of β.

We now discuss the eigenvalues for the non-uniform flows. In the range $-1 < \beta < \beta_0 = -0{\cdot}1988$, i.e. on branch C, we use numerical methods to establish the sign of the lowest eigenvalue. The results are shown in figure 4 and in table 1 and are that only for $-1 < \beta < -0{\cdot}5$ is $\lambda_1 > 0$. For $\beta_0 < \beta < 0$ and for branch B we know from our general considerations for $f(\eta) < 0$ in some range of η that sequences of negative eigenvalues exist.

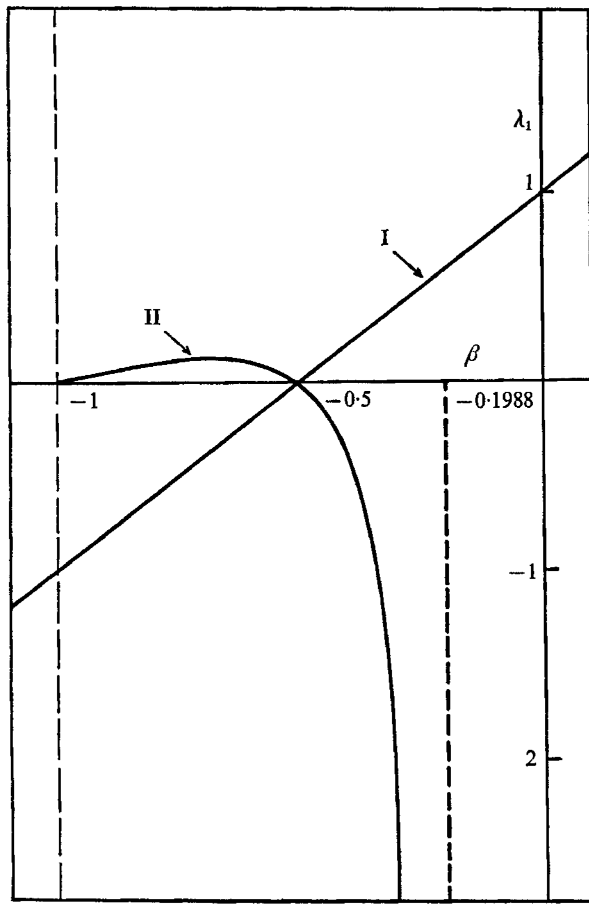

FIGURE 4. The lowest eigenvalues for the trivial and non-trivial solution branch for $-1 < \beta < \beta_0$. I, λ_1 for the trivial solution; II, λ_1 for the non-trivial solution.

As to the first new solution branch going through the point $f'(0) = 0$ for $\beta = -1{\cdot}347$, we know that the solutions for $f'(0) < 0$ and $f'(0) > 3{\cdot}5$ are spatially unstable because of the negative f' in some regions $\eta > 0$. In the range $0 < f'(0) < 3{\cdot}5$ it is found that there exist one or more negative eigenvalues preceded by an infinite sequence of positive λ_n's. Therefore, the entire branch is unstable.

However, the second lowest eigenvalue of this branch would behave almost like the lowest one of the conventional branch, i.e. $\lambda_2 \geqslant 0$ between the points $(\beta = -1{\cdot}5, f'(0) = 1)$ and $(\beta = -1{\cdot}556, f'(0) = 2{\cdot}21)$, while $\lambda_2 < 0$ over the other regions. Physically, we may argue that if we can suppress the lowest mode from the initial profile the solutions of this branch with $\lambda_2 > 0$ can be considered to be spatially stable.

The same phenomena occur on the second new branch, i.e. on branch E, $\lambda_1, \lambda_2 < 0$ over all regions and $\lambda_3 \geqslant 0$ only between the points

$$(\beta = -2{\cdot}5, f'(0) = 1) \quad \text{and} \quad (\beta = -2{\cdot}5002, f'(0) = 0{\cdot}94).$$

Thus, we have examined the linearized spatial stability of all the presently available wake-like and jet-like solutions of the Falkner–Skan equation and find that the uniform flow $f'(\eta) \equiv 1$ is stable for $\beta > -0{\cdot}5$, and that the non-uniform flow for $-1 < \beta < -0{\cdot}5$ is stable; all other flows are spatially unstable.

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